A NOTE ON PROPERLY DISCONTINUOUS ACTIONS

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This note is meant to clarify the relation between different commonly used definitions of proper discontinuity without the local compactness assumption for the underlying topological space. Much of the discussion applies to actions of nondiscrete topological groups, but, since my primary interest is geometric group theory, I will work only with discrete groups. All group actions are assumed to be continuous, in other words, these are homomorphisms from abstract groups to groups of homeomorphisms of topological spaces. This combination of *continuous* and *properly discontinuous*, sadly, leads to the ugly terminology "a continuous properly discontinuous action." A better terminology might be that of a *properly discrete* action, since it refers to proper actions of discrete groups.

Throughout this note, I will be working only with topological spaces which are 1st countable, since spaces most common in metric geometry, geometric topology, algebraic topology and geometric group theory satisfy this property. One advantage of this assumption is that if (x_n) is a sequence converging to a point $x \in X$, then the subset $\{x\} \cup \{x_n : n \in \mathbb{N}\}$ is compact, which is not true if we work with nets instead of sequences. However, I will try to avoid the local compactness assumption whenever possible, since many spaces appearing in metric geometry and geometric group theory (e.g. asymptotic cones) and algebraic topology (e.g. CW complexes) are not locally compact. (Recall that topological space X is *locally compact* if every point has a basis of topology consisting of relatively compact subsets.)

Lastly, all the material of this note is elementary and by no means original.

I ma grateful to Boris Okun for pointing out several typos and the reference to [7].

1. Group actions

A *topological group* is a group G equipped with a topology such that the multiplication and inversion maps

$$G \times G \to G, (g, h) \mapsto gh, G \to G, g \mapsto g^{-1}$$

are both continuous. A *discrete group* is a group with discrete topology. Every discrete group is clearly a topological group.

A *left continuous action* of a topological group G on a topological space X is a continuous map

$$\lambda : G \times X \to X$$

satisfying

1. $\lambda(1_G, x) = x$ for all $x \in X$.

Date: November 19, 2020.

2. $\lambda(gh, x) = \lambda(g, \lambda(h, x))$, for all $x \in X$, $g, h \in G$.

From this, it follows that the map $\rho : G \to Homeo(X)$

$$\rho(g)(x) = \lambda(g, x),$$

is a group homomorphism, where the group operation $\phi\psi$ on $Homeo(X)$ is the composition $\phi \circ \psi$.

If G is discrete, then every homomorphism $G \to Homeo(X)$ defines a left continuous action of G on X.

The shorthand for $\rho(g)(x)$ is gx or $g \cdot x$. Similarly, for a subset $A \subset X$, GA or $G \cdot A$, denotes the *orbit of* A under the G-action:

$$GA = \bigcup_{g \in G} gA.$$

The *quotient space* X/G (also frequently denoted $G\backslash X$), of X by the G-action, is the set of G-orbits of points in X, equipped with the quotient topology: The elements of X/G are equivalence classes in X, where $x \sim y$ when $Gx = Gy$ (equivalently, $y \in Gx$).

The *stabilizer* of a point $x \in X$ under the G-action is the subgroup $G_x < G$ given by

$$\{g \in G : gx = x\}.$$

An action of G on X is called *free* if $G_x = \{1\}$ for all $x \in X$. Assuming that X is Hausdorff, G_x is closed in G for every $x \in X$.

Example 1. *An example of a left action of G is the action of G on itself via left multiplication:*

$$\lambda(g, h) = gh.$$

In this case, the common notation for $\rho(g)$ is L_g. This action is free.

2. Proper maps

Properness of certain maps is the most common form of defining proper discontinuity; sadly, there are two competing notions of properness in the literature.

A continuous map $f : X \to Y$ of topological spaces is *proper in the sense of Bourbaki*, or simply *Bourbaki–proper* (cf. [2, Ch. I, §10, Theorem 1]) if f is a closed map (images of closed subsets are closed) and point–preimages $f^{-1}(y), y \in Y$, are compact. A continuous map $f : X \to Y$ is *proper* (and this is the most common definition) if for every compact subset $K \subset X$, $f^{-1}(K)$ is compact. It is noted in [2, Ch. I, §10; Prop. 7] that if X is Hausdorff and Y is locally compact then f is Bourbaki–proper if and only if f is proper.

The advantage of the notion of Bourbaki-properness is that it applies in the case of Zariski topology, where spaces tend to be compact (every subset of a finite-dimensional affine space is Zariski-compact) and, hence, the standard notion of properness is useless.

Since our goal is to trade local compactness for 1st countability, we will prove

Lemma 2. *If $f : X \to Y$ is proper, and X, Y are Hausdorff and 1st countable, then f is Bourbaki-proper.*

Proof. We only have to verify that f is closed. Suppose that $A \subset X$ is a closed subset. Since Y is 1st countable, it suffices to show that for each sequence (x_n) in A such that $(f(x_n))$ converges to $y \in Y$, there is a subsequence (x_{n_k}) which converges to some $x \in A$ such that $f(x) = y$. The subset $C = \{y\} \cup \{f(x_n) : n \in \mathbb{N}\} \subset Y$ is compact. Hence, by properness of f, $K = f^{-1}(C)$ is also compact. Since X is Hausdorff, and K is compact, follows that (x_n) subconverges to a point $x \in K$. By continuity of f, $f(x) = y$. Since A is closed, $x \in A$. □

Remark 3. *In fact, for this lemma to hold, one does not need to assume that X is Hausdorff and 1st countable, see*

The converse (each Bourbaki–proper map is proper) is proven in [2, Ch. I, §10; Prop. 6] without any restrictions on X, Y. Hence:

Corollary 4. *For maps between 1st countable Hausdorff spaces, Bourbaki–properness is equivalent to properness.*

3. Proper discontinuity

Suppose that X is a 1st countable Hausdorff topological space, G a discrete group and $G \times X \to X$ a (continuous) action. We use the notation $g_n \to \infty$ in G to indicate that g_n converges to ∞ in the 1-point compactification $G \cup \{\infty\}$ of G, i.e. for every finite subset $F \subset G$,

$$\mathrm{card}(\{n : g_n \in F\}) < \infty.$$

Given a group action $G \times X \to X$ and two subsets $A, B \subset X$, the *transporter subset* $(A|B)_G$ is defined as

$$(A|B)_G := \{g \in G : gA \cap B \neq \emptyset\}.$$

Properness of group actions is (typically) stated using certain transporter sets.

Definition 5. *Two points $x, y \in X$ are said to be G-*dynamically related *if there is a sequence $g_n \to \infty$ in G and a sequence $x_n \to x$ in X such that $g_n x_n \to y$.*

A point $x \in X$ is said to be a *wandering point* of the G-action if there is a neighborhood U of x such that $(U|U)_G$ is finite.

Lemma 6. *Suppose that the action $G \times X \to X$ is wandering at a point $x \in X$. Then the G-action has a G-*slice *at x, i.e. a neighborhood $W_x \subset U$ which is G_x-stable and for all $g \notin G_x$, $gW_x \cap W_x = \emptyset$.*

Proof. For each $g \in (U|U)_G - G_x$ we pick a neighborhood $V_g \subset U$ of x such that

$$gV_g \cap V_g = \emptyset.$$

Then the intersection

$$V := \bigcap_{g \in (U|U)_G - G_x} V_g$$

satisfies the property that $(V|V)_G = G_x$. Lastly, take

$$W_x := \bigcap_{g\in G_x} V.$$

$\square$

The next lemma is clear:

Lemma 7. *Assuming that X is Hausdorff and 1st countable, the action $G \times X \to X$ is wandering at x if and only if x is not dynamically related to itself.*

Given a group action $\alpha : G \times X \to X$, we have the natural map

$$\hat{\alpha} := \alpha \times \mathrm{id}_X : G \times X \to X \times X$$

where $\mathrm{id}_X : (g, x) \mapsto x$.

Definition 8. *An action α of a discrete group G on a topological space X is* Bourbaki–proper *if the map $\hat{\alpha}$ is Bourbaki-proper.*

Lemma 9. *If the action $\alpha : G \times X \to X$ of a discrete group G on an Hausdorff topological space X is Bourbaki-proper then the quotient space X/G is Hausdorff.*

Proof. The quotient map $X \to X/G$ is an open map by the definition of the quotient topology on X/G. Since α is Bourbaki-proper, the image of the map $\hat{\alpha}$ is closed in $X \times X$. This image is the equivalence relation on $X \times X$ which use used to form the quotient X/G. Now, Hausdorffness of X/G follows from Theorem 7.7 in [9]. $\square$

Definition 10. *An action α of a discrete group G on a topological space X is* properly discontinuous *if the map $\hat{\alpha}$ is proper.*

We note that equivalence of (1) and (5) in the following theorem is proven in [2, Ch. III, §4.4, Proposition 7] without any assumptions on X.

Theorem 11. *Assuming that X is Hausdorff and 1st countable, the following are equivalent:*

(1) *The action $\alpha : G \times X \to X$ is Bourbaki-proper.*
(2) *For every compact subset $K \subset X$,*

$$\mathrm{card}((K|K)_G) < \infty.$$

(3) *The action $\alpha : G \times X \to X$ is proper, i.e. the map $\hat{\alpha}$ is proper.*
(4) *For every compact subset $K \subset X$, there exists an open neighborhood U of K such that* $\mathrm{card}((U|U)_G) < \infty$.
(5) *For any pair of points $x, y \in X$ there is a pair of neighborhoods U_x, V_x (of x, y respectively) such that* $\mathrm{card}((U_x|V_y)_G)) < \infty$.
(6) *There are no G-dynamically related points in X.*
(7) *Assuming, that G is countable and X is completely metrizable*[1] *: The G-stabilizer of every $x \in X$ is finite and for any two points $x \in X, y \in X - Gx$, there exists a pair of neighborhoods U_x, V_y (of x, resp. y) such that $\forall g \in G$, $gU_x \cap V_y = \emptyset$.*

[1]It suffices to assume that X is *hereditarily Baire*: Every closed subset of X is Baire.

(8) *Assuming that X is a metric space and the action $G \times X \to X$ is equicontinuous*[2]*: There is no $x \in X$ and a sequence $h_n \to \infty$ in G such that $h_n x \to x$.*
(9) *Assuming that X is a metric space and the action $G \times X \to X$ is equicontinuous: Every $x \in X$ is a wandering point of the G-action.*
(10) *Assuming that X is a CW complex and the action $G \times X \to X$ is cellular: Every point of X is wandering.*
(11) *Assuming that X is a CW complex the action $G \times X \to X$ is cellular: Every cell in X has finite G-stabilizer.*

Proof. The action α is Bourbaki-proper if and only if the map $\hat{\alpha}$ is proper (see Corollary 4) which is equivalent to the statement that for each compact $K \subset X$, the subset $(K|K)_G \times K$ is compact. Hence, (1) $\iff$ (2).

The property (3) means that for each compact $K \subset X$, $\hat{\alpha}^{-1}(K \times K) = \{(g, x) \in G \times K : x \in K, gx \in K\}$ is compact. This subset is closed in $G \times X$ and projects onto $(K|K)_G$ in the first factor and to the subset

$$\bigcup_{g \in (K|K)_G} g^{-1}(K). \tag{12}$$

Hence, properness of α implies finiteness of $(K|K)_G$. Conversely, if $(K|K)_G$ is finite, compactness of $g^{-1}(K)$ for every $g \in G$ implies finiteness of the union (12). Thus, (2) $\iff$ (3).

In order to show (2)$\Rightarrow$(6), suppose that x, y are G-dynamically related points: There exists an sequence $g_n \to \infty$ in G and a sequence $x_n \to x$ such that $g_n(x_n) \to y$. The subset

$$K = \{x, y\} \cup \{x_n, g_n(x_n) : n \in \mathbb{N}\}$$

is compact. However, $y_n \in g_n(K) \cap K$ for every n. A contradiction.

(6)$\Rightarrow$(5): Suppose that the neighborhoods U_x, V_y do not exist. Let $\{U_n\}_{n \in \mathbb{N}}$, $\{V_n\}_{n \in \mathbb{N}}$ be countable bases at x, y respectively. Then for every n there exists $g_n \in G$, such that $g_n(U_n) \cap V_n \neq \emptyset$ for infinitely many g_n's in G. After extraction, $g_n \to \infty$ in G. This yields points $x_n \in U_n, y_n = g_n(x_n) \in V_n$. Hence, $x_n \to x, y_n \to y$. Thus, x is G-dynamically related to y. A contradiction.

(5)$\Rightarrow$(4). Consider a compact $K \subset X$. Then for each $x \in K, y \in K$ there exist neighborhoods U_x, V_y such that $(U_x|V_y)_G$ is finite. The product sets $U_x \times V_y, x, y \in K$ constitute an open cover of K^2. By compactness of K^2, there exist $x_1, ..., x_n, y_1, ..., y_m \in K$ such that

$$K \subset U_{x_1} \cup ... \cup U_{x_n}$$

$$K \subset V_{y_1} \cup ... \cup V_{y_m}$$

and for each pair (x_i, y_j),

$$\mathrm{card}(\{g \in G : gU_{x_i} \cap V_{y_j} \neq \emptyset\}) < \infty.$$

[2] e.g. an isometric action

Setting

$$W := \bigcup_{i=1}^{n} U_{x_i}, V := \bigcup_{j=1}^{m} V_{y_j}$$

we see that

$$\mathrm{card}((W|V)_G) < \infty.$$

Taking $U := V \cap W$ yields the required subset U.

The implication (4)⇒(2) is immediate.

Thus, we concluded the proof of equivalence of the properties (1)—(6).

(5)⇒(7): Finiteness of G-stabilizers of points in X is clear. Let x, y be points in distinct G-orbits. Let U'_x, V'_y be neighborhoods of x, y such that $(U'_x|V'_y)_G = \{g_1, ..., g_n\}$. For each i, since X is Hausdorff, there are disjoint neighborhoods V_i of y and W_i of $g_i(x_i)$. Now set

$$V_y := \bigcap_{i=1}^{n} V_i, \quad U_x := \bigcap_{i=1}^{n} g_i^{-1}(W_i).$$

Then $gU_x \cap V_y = \emptyset$ for every $g \in G$.

(7)⇒(6): It is clear that (7) implies that there are no dynamically related points with distinct G-orbits. In particular, every G-orbit in X is closed.

Assume now that X is completely metrizable and G is countable. Suppose that a point $x \in X$ is G-dynamically related to itself. Since the stabilizer G_x is finite, the point x is an accumulation point of Gx; moreover, Gx is closed in X. Hence, Gx is a closed perfect subset of X. Since X admits a complete metric, so does its closed subset Gx. Thus, for each $g \in G$, the complement $U_g := Gx - \{gx\}$ is open and dense in Gx. By the Baire Category Theorem, the countable intersection

$$\bigcap_{g \in G} U_g$$

is dense in Gx. However, this intersection is empty. A contradiction.

It is clear that (6)⇒(8) (without any extra assumptions).

(8)⇒(6). Suppose that X is a metric space and the G-action is equicontinuous. Equicontinuity implies that for each $z \in X$, a sequence $z_n \to z$ and $g_n \in G$,

$$g_n z_n \to gz.$$

Suppose that there exist a pair of G-dynamically related points $x, y \in X$: $\exists x_n \to x, g_n \in G$, $g_n x_n \to y$. By equicontinuity of the action, $g_n x \to y$. Since $g_n \to \infty$, there exist subsequences $g_{n_i} \to \infty$ and $g_{m_i} \to \infty$ such that the products $h_i := g_{n_i}^{-1} g_{m_i}$ are all distinct. Then, by equicontinuity,

$$h_i x \to x.$$

A contradiction.

The implications (5)⇒(9)⇒(8) and (5)⇒(10)⇒(11) are clear.

Lastly, let us prove the implication (11)⇒(2). We first observe that every CW complex is Hausdorff and 1st countable. Furthermore, every compact $K \subset X$ intersects only finitely many open cells e_λ in X. (Otherwise, picking one point from each nonempty intersection $K \cap e_\lambda$ we obtain an infinite closed discrete subset of K.) Thus, there exists a finite subset $E := \{e_\lambda : \lambda \in \Lambda\}$ of open cells in X such that for every $g \in (K|K)_G$, $gE \cap E \neq \emptyset$. Now, finiteness of $(K|K)_G$ follows from finiteness of cell-stabilizers in G. $\square$

Unfortunately, the property that every point of X is a wandering point is frequently taken as the definition of proper discontinuity for G-actions, see e.g. [3]. Items (8) and (10) in the above theorem provide a (weak) justification for this abuse of terminology. I feel that the better name for such actions is *wandering actions*.

Example 13. *Consider the action of $G = \mathbb{Z}$ on the punctured affine plane $X = \mathbb{R}^2 - \{(0,0)\}$, where the generator of $\mathbb{Z}$ acts via $(x,y) \mapsto (2x, \frac{1}{2}y)$. Then for any $p \in X$, the G-orbit Gp has no accumulation points in X. However, any two points $p = (x,0), q = (0,y) \in X$ are dynamically related. Thus, the action of G is not properly discontinuous.*

This example shows that the quotient space of a wandering action need not be Hausdorff.

Lemma 14. *Suppose that $G \times X \to X$ is a wandering action. Then each G-orbit is closed and discrete in X. In particular, the quotient space X/G is T1.*

Proof. Suppose that Gx accumulates at a point y. Then $Gx \cap W_y$ is nonempty, where W_y is a G-slice at y. It follows that all points of $Gx \cap W_y$ lie in the same W_y-orbit, which implies that $Gx \cap W_y = \{y\}$. $\square$

There are several reasons to consider properly discontinuous actions; one reason is that such actions yield *orbi-covering maps* in the case of smooth group actions on manifolds: $M \to M/G$ is an orbi-covering provided that the action of G on M is smooth (or, at least, locally smoothable). Another reason is that for a properly discontinuous action on a Hausdorff space, $G \times X \to X$, the quotient X/G is again Hausdorff.

Question 15. *Suppose that G is a discrete group, $G \times X \to X$ is a free continuous action on an n-dimensional topological manifold X such that the quotient space X/G is a (Hausdorff) n-dimensional topological manifold. Does it follow that the action of G on X is properly discontinuous?*

The answer to this question is negative if we merely assume that X is a Hausdorff topological space and X/G is Hausdorff.

4. Cocompactness

There are two common notions of cocompactness for group actions:

(1) $G \times X \to X$ is cocompact if there exists a compact $K \subset X$ such that $G \cdot K = X$.
(2) $G \times X \to X$ is cocompact if X/G is compact.

It is clear that (1)$\Rightarrow$(2), as the image of a compact under the continuous (quotient) map $p : X \to X/G$ is compact.

Lemma 16. *If X is locally compact then (2)$\Rightarrow$(1).*

Proof. For each $x \in X$ let U_x denote a relatively compact neighborhood of x in X. Then

$$V_x := p(U_x) = p(G \cdot U_x),$$

is compact since $G \cdot U_x$ is open in X. Thus, we obtain an open cover $\{V_x : x \in X\}$ of X/G. Since X/G is compact, this open cover contains a finite subcover

$$V_{x_1}, ..., V_{x_n}.$$

It follows that

$$p(\bigcup_{i=1}^{n} U_{x_i}) = X/G.$$

The set

$$K = \bigcup_{i=1}^{n} \overline{U_{x_i}}$$

is compact and $p(K) = X/G$. Hence, $G \cdot K = X$. □

Lemma 17. *Suppose that X is normal, $G \times X \to X$ is a proper action such that X/G is locally compact. Then X is locally compact.*

Proof. Pick $x \in X$. Let W_x be a slice for the G-action at x; then $W_x/G_x \to X/G$ is a topological embedding. Thus, our assumptions imply that W_x/G_x is compact for every $x \in X$. Let (x_α) be a net in W_x. Since W_x/G_x is compact, the net $(x_\alpha)/G$ contains a convergent subnet. Thus, after passing to a subnet, there exists $g \in G_x$ such that (gx_α) converges to some $x \in \overline{W_x}$. Hence, (x_α) subconverges to $g^{-1}(x)$. Thus, W_x is relatively compact. Since X is assumed to be normal, x admits a basis of relatively compact neighborhoods. □

5. Fundamental sets

Definition 18. *A closed subset $F \subset X$ is a* fundamental set *for the action of G on X if $G \cdot F = X$ and there exists an open neighborhood $U = U_F$ of F such that for every compact $K \subset X$, the transporter set $(U|K)_G$ is finite (the* local finiteness condition*).*

Fundamental sets appear naturally in the reduction theory of arithmetic groups (Siegel sets), see [8] and [1].

For each fundamental set F we define its quotient space F/G as the quotient space of the equivalence relation $x \sim y \iff (\{x\}|\{y\})_G \neq \emptyset$.

Lemma 19. *Suppose that F is a fundamental set for a proper action of G on a 1st countable and Hausdorff space. Then the natural projection map $p : F/G \to X/G$ is a homeomorphism.*

Proof. The map p is continuous by the definition of the quotient topology. It is also obviously a bijection. It remains to show that p is a closed map. Since F is closed, it suffices to show that the projection $q : F \to X/G$ is a closed map. Suppose that (x_n) is a sequence in F such that $q(x_n)$ converges to some $y \in X/G$, y is represented by a point $x \in F$. Then there is a sequence $g_n \in G$ such that $g_n(x_n)$ converges to x. Since $\{g_n(x_n) : n \in \mathbb{N}\} \cup \{x\}$ is compact which, without loss of generality is contained in U_F, the local finiteness assumption implies that the sequence (g_n) is finite. Hence, after extraction, $g_n = g$ for all n. The fact that F is closed then implies that $x \in F$. It follows that x is an accumulation point of (x_n). Thus, $q : F \to F/G$ is a closed map. $\square$

The next lemma, proven in [6, Lemma 2], guarantees existence of fundamental sets under the *paracompactness assumption* on X/G.

Lemma 20. *Each properly discontinuous action $G \curvearrowright X$ with paracompact quotient X/G admits a fundamental set.*

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